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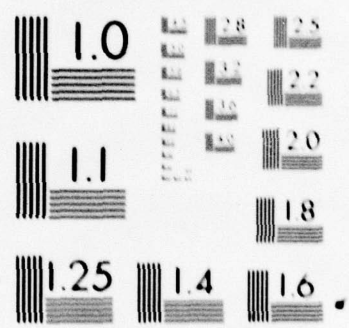


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# CIRCULAR BALANCED REPEATED MEASUREMENTS DESIGNS.

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by

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<u>Key words and phrases</u>: Repeated measurements designs, circular strongly balanced uniform designs, residual effects, universal optimality.

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# CIRCULAR BALANCED REPEATED MEASUREMENTS DESIGNS

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## ABSTRACT

The concept of a circular design is defined and when proper balance for various effects is assumed, its universal optimality is proved over the class of all designs with the same set of parameters. Such designs are shown to minimize the variance of the best linear unbiased estimators of contrasts of residual and direct effects over the class of equireplicated designs. All models assume first order residual effects and are of a circular nature.



Key words and phrases: Repeated measurements designs, circular strongly balanced uniform designs, residual effects, universal optimality.

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## 1. Introduction.

An experiment in which a unit is exposed repeatedly to a sequence of treatments is called a repeated measurements design (RMD). The experiment is based on $t$ treatments, $n$ experimental units and $p$ periods, each unit being given one treatment during each period. A repeated measurements design can therefore be thought of as a $p \times n$ array, $d$, with treatments as entries. We label the treatment $1,2,\ldots,t$. The $(i,j)^{th}$ entry in $d$ is referred to by $d(i,j)$, i.e., $d(i,j)$ is the treatment administered during the $i^{th}$ period to the $j^{th}$ experimental unit. The response models that we assume here differ somewhat from the usual models with residual effects (see Hedayat and Afsarinejad (1975) and (1978), Magda and Hedayat (1979), or Cheng and Wu (1979) in that they assume residual effects even during the first period. Our first model is as follows: an observation $Y_{ij}$, taken in row $i$ and column $j$ of $d$, can be expressed as

$$Y_{ij} = \alpha_i+\beta_j + \tau_{d(i,j)} + \rho_{d(i-1,j)} + \epsilon_{ij};\ 1 \leq i \leq p,\ 1 \leq j \leq n \quad (1.1)$$

where $\epsilon_{ij}$ are uncorrelated random variables with common variance and zero means. The unknown constants $\alpha_i$, $\beta_j$, $\tau_{d(i,j)}$ and $\rho_{d(i-1,j)}$ are respectively called the $i^{th}$ period effect, $j^{th}$ experimental unit effect, $d(i,j)^{th}$ direct treatment effect and $d(i-1,j)^{th}$ (first order) residual treatment effect. We assume that for $i = 1$ $\rho_{d(0,j)} = \rho_{d(p,j)}$, i.e., the residuals in the first period are incurred from the last period. This way each period contains residuals from the immediately preceding period,

assuming for that matter that the last period precedes the first.

With this assumption one is inclined to think of the columns of d being circular and hence also call these designs circular. Another model that we consider is the same as (1.1) but no period effect is present, i.e.,

$$Y_{ij} = \beta_j + \tau_{d(i,j)} + \rho_{d(i-1,j)} + \epsilon_{ij};\ 1 \leq i \leq p,\ 1 \leq j \leq n. \quad (1.2)$$

Similarly, the model may not contain unit effects,

$$Y_{ij} = \alpha_i + \tau_{d(i,j)} + \rho_{d(i-1,j)} + \epsilon_{ij};\ 1 \leq i \leq p,\ 1 \leq j \leq n. \quad (1.3)$$

When neither period nor unit effects are present the model assumed is:

$$Y_{ij} = \tau_{d(i,j)} + \rho_{d(i-1,j)} + \epsilon_{ij};\ 1 \leq i \leq p,\ 1 \leq j \leq n. \quad (1.4)$$

In all these models the pn observations are taken on a pxn array d in which the columns are assumed circular, as described above. Residuals, being the principal effects, are assumed in all the models. Each of the last three models can very often be associated with a sequence of observations on the same unit (or period) rather than a two dimensional array. The case with $n=1$ can be particularly meaningful. When $n > 1$ resting periods for the unit are allowed after each set of p treatments. More details on the interpretation of the above models can be found in Magda (1979).

Let us say a few words about the circular nature of the columns. One can certainly encounter practical situations where

blocks are of such nature that residual effects are transmitted along one direction in the block in a circular manner. But even if the natural setting of the experiment is such that no residuals would appear during the first period (as is assumed in most papers on the subject) one can correct that in the following way. Introduce a pre-period (or period 0) in which treatments are given to the units only to be used for residuals during period 1. No observations need to be taken during period 0. But it is essential that during the last period of the experiment the same distribution of treatments as in period 0 will be made to the units. This way the residuals from the last period (being the same as those in period 0) will be observed during period 1.

Balancing the experiment simultaneously for residuals, periods and units becomes very natural under this model.

The concept of strongly balanced uniform RMD has recently been brought up by Cheng and Wu (1979) in the model with no residual effects during the first period. Necessary divisibility conditions for the existence of such a design are (for $\lambda_1$, $\lambda_2$ and $\lambda$ positive integers):

$$p = \lambda_1 t, \; n = \lambda_2 t \quad \text{and} \quad \lambda = \frac{(p-1)n}{t^2} .$$

These three conditions imply $t^2 | n$ which means that the number of experimental units can get rather large. In the case of a circular strongly balanced uniform RMD (which we shall define shortly)

$$p = \lambda_1 t \quad \text{and} \quad n = \lambda_2 t$$

turn out to be the only necessary divisibility conditions.

## 2. Definitions.

Let us recall the usual notations and definitions.

A repeated measurements design based on $t$ treatments, $n$ experimental units and $p$ periods will be abbreviated by RMD$(t,n,p)$. The collection of all such designs is denoted by $\Omega_{t,n,p}$.

Definition 2.1. A design is called uniform on periods if each treatment occurs the same number of times, say $\lambda_1$, in each period.

A necessary condition for this to happen is $n = \lambda_1 t$.

Definition 2.2. A design is said to be uniform on units if each treatment is assigned the same number of times, say $\lambda_2$, to each unit.

This can occur only if $p = \lambda_2 t$.

Definition 2.3. A design is called uniform if it is uniform on both periods and units.

Definition 2.4. A design $d \in \Omega_{t,n,p}$ is called circular strongly balanced if the collection of ordered pairs $(d(i,j), d(i+1,j))$, $1 \leq i \leq p$, $1 \leq j \leq n$ contains each ordered pair of treatments (distinct or not) the same number of times, say $\lambda$.

When $i = p$, $i+1 = 1$, because we also count the pairs $(d(p,j), d(1,j))$, since the columns are assumed circular.

A circular strongly balanced design can exist only if $np = \lambda t^2$.

It is not hard to check that in a circular strongly balanced uniform RMD the following divisibility relations hold:

$$p = \lambda_1 t, \quad n = \lambda_2 t \quad \text{and} \quad \lambda = \lambda_1 \lambda_2.$$

Hence $\lambda$ is always integral if $\lambda_1$ and $\lambda_2$ are.

Example 2.1. The following is an example of a circular strongly balanced uniform RMD(4,4,8):

$$d^* = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \\ 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{array} \qquad (\lambda = 2).$$

In the above example we were able to construct a circular strongly balanced uniform design with 4 treatments, based on only 4 units. To achieve a similar balance on the 4 units when no residual effects are assumed in the first period takes at least 16 units. (See the model and the definition of a strongly balanced uniform RMD in Cheng and Wu (1979).

Definition 2.5. A design $d \in \Omega_{t,n,p}$ is called circular balanced if the collection of ordered pairs $(d(i,j), d(i+1,j))$, $1 \leq i \leq p$,

$1 \leq j \leq n$ contains each ordered pair of distinct treatments $\lambda$ times.

As before, the columns of $d$ are assumed circular.

The divisibility condition for the existence of a circular balanced RMD is: $pn = \lambda t(t-1)$.

Example 2.2.

$$d = \begin{array}{ccccccc} 1&2&3&4&5&6&7 \\ 2&3&4&5&6&7&1 \\ 4&5&6&7&1&2&3 \end{array} \quad \begin{array}{ccccccc} 1&2&3&4&5&6&7 \\ 4&5&6&7&1&2&3 \\ 2&3&4&5&6&7&1 \end{array}$$

$d \in \Omega_{7,14,3}$ is a circular balanced RMD uniform on periods.

3. Optimality of some circular balanced RMD's.

The first two lemmas which we shall prove have easy, but important statistical interpretations. They state that for the purpose of estimating linear functions of certain parameters, we can only decrease the number of such estimable functions and (in connection with Lemma 2.1 of Ehrenfeld (1955)) only decrease the precision of our estimators by allowing more parameters in the model.

Under the general linear model

$$E(Y) = X_1\theta_1 + X_2\theta_2 + \ldots + X_r\theta_r, \quad \text{cov } Y = \sigma^2 I$$

we denote by $C_r(\theta_1)$ the information matrix of $\theta_1$.

**Lemma 3.1.** For the two linear models

$$E(Y) = \sum_{i=1}^{k} X_i\theta_i \quad \text{and} \quad E(Y) = \sum_{i=1}^{n} X_i\theta_i, \quad \text{cov } Y = \sigma^2 I$$

with $k \leq n$ we have $C_n(\theta_1) \leq C_k(\theta_1)$.

**Proof:** For convenience, let us write $X_i^{(1)}$ for $X_i$, $1 \leq i \leq n$, and reorder the parameters in the design matrix (1) as follows:

$$^{(1)} = [X_1^{(1)}, X_n^{(1)}, X_{n-1}^{(1)}, \ldots, X_3^{(1)}, X_2^{(1)}].$$

If $\theta_1$ is a vector of length $s$, the $s{\times}s$ matrix in the upper left hand corner of $X^{(1)\prime}X^{(1)}$ is $C_1(\theta_1) = X_1^{(1)\prime}X_1^{(1)}$. Do the following operation in $X_1^{(1)\prime}X_1^{(1)} = (X_i^{(1)\prime}X_j^{(1)})$, $i,j = 1,n,n-1,\ldots,3,2$. Form the matrix $X^{(2)\prime}X^{(2)}$ as follows:

$$X^{(2)'}X^{(2)} = \begin{bmatrix} X_1^{(1)'}X_1^{(1)} & X_1^{(1)'}X_n^{(1)} & X_1^{(1)'}X_{n-1}^{(1)} & \cdots & X_1^{(1)'}X_3^{(1)} \\ X_n^{(1)'}X_1^{(1)} & X_n^{(1)'}X_n^{(1)} & X_n^{(1)'}X_{n-1}^{(1)} & \cdots & X_n^{(1)'}X_3^{(1)} \\ & & \cdots & & \\ X_3^{(1)'}X_1^{(1)} & X_3^{(1)'}X_n^{(1)} & X_3^{(1)'}X_{n-1}^{(1)} & \cdots & X_3^{(1)'}X_3^{(1)} \end{bmatrix} -$$

$$- \begin{bmatrix} X_1^{(1)'}X_2^{(1)} \\ X_n^{(1)'}X_2^{(1)} \\ X_{n-1}^{(1)'}X_2^{(1)} \\ \vdots \\ X_3^{(1)'}X_2^{(1)} \end{bmatrix} (X_2^{(1)'}X_2^{(1)})^{-}[X_2^{(1)'}X_1^{(1)}, X_2^{(1)'}X_n^{(1)}, X_2^{(1)'}X_{n-1}^{(1)}, \ldots, X_2^{(1)'}X_3^{(1)}]$$

where $(X_2^{(1)'}X_2^{(1)})^-$ is a generalized inverse of $X_2^{(1)'}X_2^{(1)}$. Explicitly, $X^{(2)} = [X_1^{(2)}, X_{n-1}^{(2)}, X_{n-2}^{(2)}, \ldots, X_3^{(2)}, X_2^{(2)}]$, where

$$X_i^{(2)} = (I - X_2^{(1)}(X_2^{(1)'}X_2^{(1)})^- X_2^{(1)'})X_{i+1}^{(1)}, \quad 2 \le i \le n-1$$

and $X_1^{(2)} = (I-(X_2^{(1)}(X_2^{(1)'}X_2^{(1)})^- X_2^{(1)'})X_1^{(1)}$. Then in the same sxs upper left hand corner we have

$$C_2(\theta_1) = X_1^{(2)\prime} X_1^{(2)} = X_1^{(1)\prime} X_1^{(1)} -$$

$$- X_1^{(1)\prime} X_2^{(1)} ( X_2^{(1)\prime} X_2^{(1)} )^- X_2^{(1)\prime} X_1^{(1)} \leq X_1^{(1)\prime} X_1^{(1)} = C_1(\theta_1).$$

We can now proceed recursively. By the same kind of elimination in $X^{(r)}$ we obtain:

$$C_{r+1}(\theta_1) = X_1^{(r+1)\prime} X_1^{(r+1} = X_1^{(r)\prime} X_1^{(r)} - X_1^{(r)\prime} X_2^{(r)} ( X_2^{(r)\prime} X_2^{(r)} )^- X_2^{(r)\prime} X_1^{(r)} \leq$$

$$\leq X_1^{(r)\prime} X_1^{(r)} = C_r(\theta_1)$$ and conclude that $C_{r+1}(\theta_1) \leq C_r(\theta_1)$,

for all $1 \leq r \leq n-1$. Hence $C_n(\theta_1) \leq C_k(\theta_1)$ for $1 \leq k \leq n$.

By the duality which exists between residual and direct effects in all the models described in section 1, it is clear that they both share the same information matrix for a fixed model. We denote by $C_d^{(1)}$, $C_d^{(2)}$, $C_d^{(3)}$, and $C_d^{(4)}$ the information matrix for residual effects when design $d$ is used under models (1.1), (1.2), (1.3) and (1.4). Since model (1.i) contains more parameters then (1.4) for $1 \leq i \leq 4$, Lemma 3.1 implies

$$C_d^{(i)} \leq C_d^{(4)} \quad \text{for} \quad 1 \leq i \leq 4 \tag{3.1}$$

We shall rely on this fact for a unified proof of several results.

The results we prove concern the universal optimality of certain circular balanced designs. The concept of universal optimality has been introduced by Kiefer (1975). It is known that if a design is universally optimal then it is D-, A- and E-optimal. Using the sufficient conditions established by Kiefer (1975) to ensure universal optimality, a design $d^*$ is universally optimal for residual effects in $\Omega_{t,n,p}$ under model (1.i) if

(i) For any $d \in \Omega_{t,n,p}$ its information matrix $C_d^{(i)}$ has zero row sums.

(ii) $C_{d^*}^{(i)}$ is completely symmetric, i.e., all the diagonal elements of $C_{d^*}^{(i)}$ are equal and all the off-diagonal elements of $C_{d^*}^{(i)}$ are equal.

(iii) $\max\limits_{d\in\Omega_{t,n,p}} \operatorname{tr} C_d^{(i)} \leq \operatorname{tr} C_{d^*}^{(i)}$.

We proceed in showing that (i), (ii) and (iii) are satisfied for suitable choices of $d^*$ under the various models.

In order to show that $C_d^{(i)}$ has row sums zero, for $1 \leq i \leq 4$, we first prove a lemma. This lemma can be also derived from Lemma 2.1 of Ehrenfeld (1955).

<u>Lemma 3.2</u>. Let $A, B \geq 0$.

If $A \leq B$ then the row span of $A$ is contained in the row span of $B$.

<u>Proof</u>: We can write $A = N'N$ and $B = M'M$ where $N$ and $M$ are of full row rank. Then

$$Mx = 0 \Rightarrow x'M'Mx = 0 \Rightarrow x'N'Nx = 0 \Rightarrow Nx = 0.$$

Therefore $\{x : Mx = 0\} \subset \{x : Nx = 0\}$, or the orthogonal complement of the row span of $M$ is contained in the orthogonal complement of the row span of $N$. Hence

$$(\text{row span of } M) \supset (\text{row span of } N).$$

Then the column span $M'M = B$ contains the column span of $N'N = A$. Since $A$ and $B$ are symmetric the same is true for the row spans.

Note that if the row sums of $C_d^{(4)}$ are zero, then the row span of $C_d^{(4)}$ is orthogonal to the vector $\underset{\sim}{1}$, whose entries are all 1. Since by (3.1) $C_d^{(i)} \leq C_d^{(4)}$ for $1 \leq i \leq 4$, Lemma 3.2 assures that the row span of $C_d^{(i)}$ is also orthogonal to $\underset{\sim}{1}$ since it is contained in the row span of $C_d^{(4)}$. In order to show (i), for $1 \leq i \leq 4$, it therefore suffices to show that the row sums of $C_d^{(4)}$ are zero.

It is easy to establish that $C_d^{(4)} = D_d - M_d D_d^{-1} M_d'$, with $D_d = (r_{di})$ and $M_d = (m_{dij})$ where

$r_{di}$ = the number of appearances of treatment $i$ in $d$.

$m_{dij}$ = the number of times treatment $i$ is immediately preceded (columnwise) by treatment $j$ in $d$.

The following dependencies hold for any $d \in \Omega_{t,n,p}$:

$$r_{di} = \sum_{j=1}^{t} m_{dij} = \sum_{j=1}^{t} m_{dji}$$

$$\text{and} \quad \sum_{i=1}^{t} r_{di} = pn \tag{3.2}$$

The $(i,j)^{th}$ entry of $C_d^{(4)} = D_d - M_d D_d^{-1} M_d'$ is

$$\delta_{ij} r_i - \sum_{k=1}^{t} \frac{1}{r_{dk}} m_{dik} m_{djk},$$

where $\delta_{ij} = 1$ if $i = j$ and $0$ otherwise. Indeed, the sum along the $i^{th}$ row in $C_d^{(4)}$ is (using (3.2)):

$$\sum_{j=1}^{t} (\delta_{ij} r_i - \sum_{k=1}^{t} \frac{1}{r_{dk}} m_{dik} m_{djk}) = r_{di} - \sum_{j=1}^{t} \sum_{k=1}^{t} \frac{1}{r_{dk}} m_{dik} m_{djk} =$$

$$= r_{di} - \sum_{k=1}^{t} \frac{1}{r_{dk}} m_{dik} \sum_{j=1}^{t} m_{djk} = r_{di} - \sum_{k=1}^{t} \frac{1}{r_{dk}} m_{dik} r_{dk}$$

$$= r_{di} - \sum_{k=1}^{t} m_{dik} = r_{di} - r_{di} = 0.$$

In what follows, $d_1^*$ denotes a circular strongly balanced uniform RMD, $d_2^*$ denotes a circular strongly balanced uniform on units RMD, $d_3^*$ denotes a circular strongly balanced uniform on periods RMD and finally, $d_4^*$ denotes just a circular strongly balanced RMD.

By doing the standard row operations under the four models we establish that

$$C^{(i)}_{d_i^*} = \frac{pn}{t}\,(I - \frac{1}{t}\,J) \quad \text{for} \quad 1 \leq i \leq 4 \tag{3.3}$$

where $I$ is the identity matrix of order $t$ and $J$ is the $t \times t$ matrix with all its entries equal to 1. Note, for example, that

$$C^{(i)}_{d_j^*} = \frac{pn}{t}\,(I - \frac{1}{t}\,J)$$

when $j \leq i$, but not when $j > i$.

The condition (ii) on complete symmetry is contained in (3.3). It remains to check (iii), i.e.,

$$\max_{d \in \Omega_{t,n,p}} \operatorname{tr} C^{(i)}_{d} \leq \operatorname{tr} C^{(i)}_{d_i^*} \quad \text{for} \quad 1 \leq i \leq 4.$$

This also happens to be easy to show, by making use of both (3.1) and (3.3). From (3.1) and (3.2)

$$\operatorname{tr} C^{(i)}_{d} \leq \operatorname{tr} C^{(4)}_{d} = \operatorname{tr}(D_d - M_d D_d^{-1} M_d') = pn - \sum_{j=1}^{t} \sum_{k=1}^{t} \frac{m^2_{djk}}{r_{dk}},$$

for all $1 \leq i \leq 4$. From (3.3)

$$\operatorname{tr} C^{(i)}_{d_i^*} = pn - \frac{pn}{t} \quad \text{for all} \quad 1 \leq i \leq 4.$$

It therefore suffices to show that $d_i^*$ minimizes $\sum_{j=1}^{t} \sum_{k=1}^{t} \frac{m^2_{djk}}{r_{dk}}$

over $\Omega_{t,n,p}$ for all $1 \leq i \leq 4$. Indeed, since

$$\sum_{j=1}^{t} m_{djk} = r_{dk} \quad \text{and} \quad \sum_{k=1}^{t} r_{dk} = pn$$

$$\sum_{j=1}^{t} \sum_{k=1}^{t} \frac{m_{djk}^2}{r_{dk}} = \sum_{k=1}^{t} \frac{1}{r_{dk}} \sum_{j=1}^{t} m_{djk}^2 \geq$$

$$\geq \sum_{k=1}^{t} \frac{1}{r_{dk}} t\left(\frac{r_{dk}}{t}\right)^2 = \sum_{k=1}^{t} \frac{r_{dk}}{t} = \frac{pn}{t} .$$

The inequality used above corresponds to the fact that the sum of squares of $t$ nonzero terms having constant sum is minimized when they are all equal.

Having satisfied the conditions (i), (ii), and (iii) for $1 \leq i \leq 4$ we have proved the following theorem:

Theorem 3.1. Under the model (1.i) whenever a design $d_i^*$ exists, it is universally optimal for the estimation of direct as well as residual effects over the collection of designs with the same parameters $(1 \leq i \leq 4)$.

The following stronger optimality result holds if we suitably restrict the class of competing designs:

Theorem 3.2. Under the model (1.i) whenever a design $d_i^*$ exists, it minimizes the variance of the best linear unbiased estimator of any contrast of direct effects and any contrast of residual

effects over the collection of equireplicated designs with the same parameters $(1 \leq i \leq 4)$.

Proof: A design $d \in \Omega_{t,n,p}$ is called equireplicated if $D_d = \frac{pn}{t} I$, or equivalently, if $r_{d1} = \cdots = r_{dt} = \frac{pn}{t}$. Let $\Omega^*_{t,n,p}$ denote the collection of equireplicated designs with parameters $t,n$ and $p$. By Lemma 2.1 of Ehrenfeld (1955), it is enough to show that

$$x' C_d^{(i)} x \leq x' C_{d_i^*}^{(i)} x$$

for all $d \in \Omega^*_{t,n,p}$ and all $t \times 1$ vectors $x$. We can write $x = y + \underset{\sim}{1}$, where $y' \underset{\sim}{1} = 0$. Then knowing that $C_d^{(i)}$ is symmetric and $C_d^{(i)} \underset{\sim}{1} = 0$ for all $d \in \Omega^*_{t,n,p}$, we have

$$x' C_d^{(i)} x = (y' + \underset{\sim}{1}') C_d^{(i)} (y + \underset{\sim}{1}) = y' C_d^{(i)} y$$

where $y' \underset{\sim}{1} = 0$. It suffices therefore to show that

$$y' C_d^{(i)} y \leq y' C_{d_i^*}^{(i)} y$$

for all $y$ such that $y' \underset{\sim}{1} = 0$. We know from (3.1) that

$$C_d^{(i)} \leq C_d^{(4)} = D_d - M_d D_d^{-1} M_d' \leq D_d = \frac{pn}{t} I$$

for $d \in \Omega^*_{t,n,p}$. Hence

$$y' C_d^{(i)} y \leq y' \frac{pn}{t} I y = y' \frac{pn}{t} (I - \frac{1}{t} J) y = y' C_{d_i^*}^{(i)} y,$$

since $y'Jy = 0$. This proves the theorem.

Up to this point we have seen that under various models various kinds of circular strongly balanced designs proved to be optimal. We now turn to the concept of a circular balanced RMD and show some of its optimal properties.

Let $\delta_1^*$ denote a circular balanced uniform RMD (see Definition (2.5), $\delta_2^*$ a circular balanced uniform on units RMD, $\delta_3^*$ a circular balanced uniform on periods RMD and finally, $\delta_4^*$ just a circular balanced RMD. As before, $C_d^{(i)}$ will denote the information matrix for residual (and hence also direct) treatment effects when model (1.1) is assumed. As previously we have

$$C_d^{(i)} \leq C_d^{(4)} \quad \text{for} \quad 1 \leq i \leq 4 \tag{3.4}$$

with $C_d^{(4)} = D_d - M_d D_d^{-1} M_d'$, where $D_d$ and $M_d$ are as before and the relations (3.2) hold. One shows that $C_d^{(i)}$ has row sums zero either directly or using Lemma 3.2 as we did previously. By carrying out the standard row operations it turns out that

$$C_{\delta_i^*}^{(i)} = \left(\frac{pp}{t} - \frac{\lambda}{t-1}\right)\left(I - \frac{1}{t} J\right) \quad \text{for} \quad 1 \leq i \leq 4.$$

It is now convenient to use the following Lemma in order to establish trace maximality. We just state it and omit the easy proof.

<u>Lemma 3.3</u>. Let $m_i$ be $n$ integers. The minimum of $\sum_{i=1}^{n} m_i^2$ subject to $\sum_{i=1}^{n} m_i = k$ is attained when $k - n\,\mathrm{int}\left(\frac{k}{n}\right)$ of the $m_i$ equal $1 + \mathrm{int}\left(\frac{k}{n}\right)$ and $n - k + n\,\mathrm{int}\left(\frac{k}{n}\right)$ of the $m_i$ equal $\mathrm{int}\left(\frac{k}{n}\right)$. The corresponding minimal value is

$$-n\left(\mathrm{int}\left(\frac{k}{n}\right)\right)^2 + (2k-n)\left(\mathrm{int}\left(\frac{k}{n}\right)\right) + k.$$

The symbol $\mathrm{int}\ x$ symbolizes the integral part of the real number $x$.

Turning to the maximization of the trace, by (3.4) and (3.2) it suffices to show that $\delta_4^*$ minimizes $\sum_{j=1}^{t}\sum_{k=1}^{t} \frac{m_{djk}^2}{r_{dk}}$. In this case a solution with all $m_{dij}$ equal and integral is not possible because of the present divisibility condition. Hence we have to carry out the minimization over integers, using Lemma 3.3, i.e.,

$$\sum_{k=1}^{t} m_{djk}^2 \geq -t\left(\mathrm{int}\left(\frac{r_{dk}}{t}\right)\right)^2 + 2(r_{dk} - t)\left(\mathrm{int}\left(\frac{r_{dk}}{t}\right)\right) + r_{dk}$$

$$\geq \left(\mathrm{int}\left(\frac{r_{dk}}{t}\right)\right)(r_{dk} - t) + r_{dk} \geq r_{dk}.$$

For any $d \in \Omega_{t,n,p}$ with $\lambda = 1$ we therefore have:

$$\sum_{j=1}^{t}\sum_{k=1}^{t} \frac{m_{djk}^2}{r_{dk}} = \sum_{k=1}^{t} \frac{1}{r_{dk}} \sum_{j=1}^{t} m_{djk}^2 \geq \sum_{k=1}^{t} \frac{1}{r_{dk}}\, r_{dk} \geq t,$$

with equality for $\delta_4^*$ (with $\lambda = 1$). This establishes the universal optimality of $\delta_4^*$ over $\Omega_{t,n,p}$ when $pn = t(t-1)$ (or equivalently, when $\lambda = 1$).

In case $\lambda > 1$, $\sum_{j=1}^{t} \sum_{k=1}^{t} \frac{m_{djk}^2}{r_{dk}}$ is minimized by $\delta_4^*$ only in a smaller class of designs. Let $\Omega'_{t,n,p}$ be the collection of designs $d \in \Omega_{t,n,p}$ with the property that no treatment precedes itself, i.e., $\Omega'_{t,n,p} = \{d: m_{dii} = 0,\ 1 \le i \le t\}$.

Then for any $d \in \Omega'_{t,n,p}$,

$$\sum_{j=1}^{t} \sum_{k=1}^{t} \frac{m_{djk}^2}{r_{dk}} = \sum_{k=1}^{t} \frac{1}{r_{dk}} \sum_{j=1}^{t} m_{djk}^2 \ge$$

$$\ge \sum_{k=1}^{t} \frac{1}{r_{dk}} (t-1)\left(\frac{r_{dk}}{t-1}\right)^2 = \frac{1}{t-1} \sum_{k=1}^{t} r_{dk} = \frac{np}{t-1} = \lambda t$$

with equality for $\delta_4^*$.

The optimality of $\delta_i^*$, $1 \le i \le 4$, cannot be extended to all of $\Omega_{t,n,p}$ when $\lambda > 1$. The following example shows this.

<u>Example</u>: Let $\delta_3^*$ be

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 2 | 3 |
| 2 | 3 | 1 | 3 | 1 | 2 |
| 3 | 1 | 2 | 2 | 3 | 1 |

and $d_3^*$ be

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 2 | 3 | 1 | 2 | 3 |
| 2 | 3 | 1 | 2 | 3 | 1. |

$\delta_3^*$ is a circular balanced uniform on periods RMD with $\lambda = 3$, while $d_3^*$ is a circular strongly balanced uniform on periods RMD.

The both belong to $\Omega_{3,6,3}$. By Theorem 3.2, $d_3^*$ is better than $\delta_3^*$. This shows that the optimality of $\delta_1^*$ can only be proved over a subclass of $\Omega_{t,n,p}$ when $\lambda > 1$.

Let us now state the results established above.

Theorem 3.3. Under the model (1.i) whenever a design $\delta_i^*$ with $\lambda = 1$ exists, it is universally optimal for the estimation of direct as well as residual effects over the collection of designs with the same parameters $(1 \leq i \leq 4)$.

Theorem 3.4. Whenever a design $\delta_i^* \in \Omega_{t,n,p}$ exists, it is universally optimal for the estimation of direct as well as residual effects under model (1.i) over $\Omega'_{t,n,p}$ $(1 \leq i \leq 4)$.

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The concept of a circular design is defined and when proper balance for various effects is assumed, its universal optimality is proved over the class of all designs with the same parameters. Such designs are shown to minimize the variance of the best linear unbiased estimators of contrasts of residual and direct effects over the class of equireplicated designs. All models assume first order residual effects and are of circular nature.